1. The Simplex

Let $\varepsilon \in [0,1)$ be a fixed constant. Consider the symmetric half-cube

$$R = \{(x,y,z) \in [0,2]^3 \ : \ x+y+z \le 3/2\}.$$

We form the following seven subregions:

$$\begin{aligned}
A_{xyz} &= \{(x,y,z) \in R \ : \ x+y \le y+z \le z+x \le 1-\varepsilon\},\\
B_{xyz} &= \{(x,y,z) \in R \ : \ x+y \le y+z \le 1-\varepsilon \le z+x \le 1+\varepsilon\},\\
C_{xyz} &= \{(x,y,z) \in R \ : \ x+y \le 1-\varepsilon \le y+z \le z+x \le 1+\varepsilon\},\\
D_{xyz} &= \{(x,y,z) \in R \ : \ 1-\varepsilon \le x+y \le y+z \le z+x \le 1+\varepsilon\},\\
E_{xyz} &= \{(x,y,z) \in R \ : \ x+y \le y+z \le 1-\varepsilon \le 1+\varepsilon \le z+x\},\\
F_{xyz} &= \{(x,y,z) \in R \ : \ x+y \le 1-\varepsilon \le y+z \le 1+\varepsilon \le z+x\},\\
G_{xyz} &= \{(x,y,z) \in R \ : \ x+y \le 1-\varepsilon \le 1+\varepsilon \le y+z \le z+x\},\\
H_{xyz} &= \{(x,y,z) \in R \ : \ 1-\varepsilon \le x+y \le y+z \le 1+\varepsilon \le z+x\}.
\end{aligned}$$

Further, we define permutations of these sets by permuting the three variables $\{x,y,z\}$ both in the subscript and in the right half of the definition of the set. These $6 \times 8 = 48$ sets cover R.

It turns out that computationally we want to break up the region F_{xyz} into three further subregions. Namely

$$\begin{aligned}
S_{xyz} &= \{(x,y,z) \in F_{xyz} \ : \ z \le 1/2+\varepsilon\},\\
T_{xyz} &= \{(x,y,z) \in F_{xyz} \ : \ z \ge 1/2+\varepsilon; x \ge 1/2-\varepsilon\},\\
U_{xyz} &= \{(x,y,z) \in F_{xyz} \ : \ x \le 1/2-\varepsilon\}.
\end{aligned}$$

Next, we define a function $F(x,y,z)$ (not to be confused with the F_{xyz} region) which has polynomial support on those ten pieces, along with the obvious symmetries. We assume further that $F|_{D_{xyz}} = 0$. So our function is really defined in terms of nine pieces of information. Finally, we will be assuming that $\varepsilon \in [1/4, 1/3]$, as one must use slightly different integrals in other ranges, and this seems to be the best range.

2. The I-integral

Given any function $F(x,y,z)$ defined on a 3-dimensional simplex S, we define the integral operator $I(S) = \int\int\int_S F(x,y,z)^2 \, dx \, dy \, dz$. To find $I := I(R)$ it suffices to find the part of I on each of the nine subregions we described in the previous section (we still ignore D_{xyz}), add them, and then multiply by 6.

We find on A_{xyz} that

$$I(A_{xyz}) = \int_{x=0}^{1/2-\varepsilon/2} \int_{y=0}^{x} \int_{z=x}^{1-\varepsilon-x} F|^2_{A_{xyz}} \, dz \, dy \, dx.$$

Next we compute

$$I(B_{xyz}) = \left(\int_{z=1/2-\varepsilon/2}^{1/2+\varepsilon/2} \int_{x=1-\varepsilon-z}^{z} + \int_{z=1/2+\varepsilon/2}^{1-\varepsilon} \int_{x=1-\varepsilon-z}^{1+\varepsilon-z} \right) \int_{y=0}^{1-\varepsilon-z} F|^2_{B_{xyz}} \, dy \, dx \, dz.$$

Note that in this integral representation we require $\varepsilon < 1/3$. The computation on C_{xyz} requires a bit more work, but we have

$$\begin{aligned} I(C_{xyz}) = &\left(\int_{y=0}^{1/2-3\varepsilon/2} \int_{x=y}^{y+2\varepsilon} + \int_{y=1/2-3\varepsilon/2}^{1/2-\varepsilon} \int_{x=y}^{1-\varepsilon-y} \right) \int_{z=1-\varepsilon-y}^{1+\varepsilon-x} \\ &+ \int_{y=1/2-\varepsilon}^{1/2-\varepsilon/2} \int_{x=y}^{1-\varepsilon-y} \int_{z=1-\varepsilon-y}^{3/2-x-y} F|^2_{C_{xyz}} \, dz \, dx \, dy. \end{aligned}$$

Next, skipping over D_{xyz}, we find

$$I(E_{xyz}) = \int_{z=1/2+\varepsilon/2}^{1-\varepsilon} \int_{x=1+\varepsilon-z}^{z} \int_{y=0}^{1-\varepsilon-z} F|^2_{E_{xyz}} \, dy \, dx \, dz.$$

The region F_{xyz} is somewhat complicated, but we broke it into three pieces. We compute

$$I(S_{xyz}) = \left(\int_{y=0}^{1/2-3\varepsilon/2} \int_{z=1-\varepsilon-y}^{1/2+\varepsilon} + \int_{y=1/2-3\varepsilon/2}^{1/2-\varepsilon} \int_{z=y+2\varepsilon}^{1/2+\varepsilon} \right) \int_{x=1+\varepsilon-z}^{1-\varepsilon-y} F|_{S_{xyz}} \, dx \, dz \, dy$$

while

$$I(T_{xyz}) = \left(\int_{z=1/2+\varepsilon}^{1/2+2\varepsilon} \int_{x=1+\varepsilon-z}^{3/2-z} + \int_{z=1/2+2\varepsilon}^{1+\varepsilon} \int_{x=1/2-\varepsilon}^{3/2-z} \right) \int_{y=0}^{3/2-x-z} F|^2_{T_{xyz}} \, dy \, dz \, dx$$

and

$$I(U_{xyz}) = \int_{x=0}^{1/2-\varepsilon} \int_{y=0}^{x} \int_{z=1+\varepsilon-x}^{1+\varepsilon-y} F|_{U_{xyz}} \, dz \, dy \, dx.$$

The G_{xyz} region contributes

$$I(G_{xyz}) = \int_{x=0}^{1/2-\varepsilon} \int_{y=0}^{x} \int_{z=1+\varepsilon-y}^{3/2-x-y} F|^2_{G_{xyz}} \, dx \, dz \, dy.$$

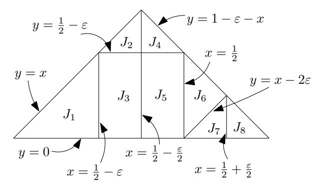
Our final region yields

$$\begin{aligned} I(H_{xyz}) = &\left(\int_{x=1/2+\varepsilon/2}^{1-\varepsilon} \int_{y=1-\varepsilon-x}^{3/2-2x} + \int_{x=1-\varepsilon}^{3/4} \int_{y=0}^{3/2-2x} \right) \int_{z=x}^{3/2-x-y} \\ &+ \int_{x=1/2}^{1/2+\varepsilon/2} \int_{y=1-\varepsilon-x}^{1/2-\varepsilon} \int_{z=1+\varepsilon-x}^{3/2-x-y} F|^2_{H_{xyz}} \, dz \, dy \, dx. \end{aligned}$$

3. The J-integral

Similar to the previous section, we define the following integral operator: $J = J^{(3)} = \int\int_{(x,y,z)\in R\,:\,x+y\le 1-\varepsilon} \left(\int_{z=0}^{\infty} F\ dz\right)^2\ dy\ dx$. This integral can only be broken up so much, due to the presence of the squaring. We will reduce to the case where $y \le x$, so our integral passes through the regions A_{xyz}, A_{yzx}, A_{zyx}, B_{xyz}, B_{zyx}, C_{xyz}, E_{xyz}, E_{zyx}, S_{xyz}, T_{xyz}, U_{xyz}, and G_{xyz}.

Projecting all intersection lines of these regions to the (x, y)-plane, we have the diagram:



The lines in this diagram, moving left to right (and then top to bottom) are $y = x$, $y = 0$, $x = 1/2 - \varepsilon$, $y = 1/2 - \varepsilon$, $x = 1/2 - \varepsilon/2$, $x = 1/2$, $y = x - 2\varepsilon$, and $x = 1/2 + \varepsilon/2$. This picture is only valid when $\varepsilon \in [1/4, 1/3]$. We label the pieces of the diagram moving from left to right and then top to bottom, and there are eight corresponding integrals $J1, J2, \ldots, J8$.

We have

$$J1 = \int_{x=0}^{1/2-\varepsilon} \int_{y=0}^{x} \left(\int_{z=0}^{y} F|_{A_{yzx}} + \int_{z=y}^{x} F|_{A_{zyx}} + \int_{z=x}^{1-\varepsilon-x} F|_{A_{xyz}} + \int_{z=1-\varepsilon-x}^{1-\varepsilon-y} F|_{B_{xyz}} \right.$$
$$\left. + \int_{z=1-\varepsilon-y}^{1+\varepsilon-x} F|_{C_{xyz}} + \int_{z=1+\varepsilon-x}^{1+\varepsilon-y} F|_{U_{xyz}} + \int_{z=1+\varepsilon-y}^{3/2-x-y} F|_{G_{xyz}}\ dz \right)^2\ dy\ dx.$$

Next comes

$$J2 = \int_{x=1/2-\varepsilon}^{1/2-\varepsilon/2} \int_{y=1/2-\varepsilon}^{x} \left(\int_{z=0}^{y} F|_{A_{yzx}} + \int_{z=y}^{x} F|_{A_{zyx}} + \int_{z=x}^{1-\varepsilon-x} F|_{A_{xyz}} + \int_{z=1-\varepsilon-x}^{1-\varepsilon-y} F|_{B_{xyz}} \right.$$
$$\left. + \int_{z=1-\varepsilon-y}^{3/2-x-y} F|_{C_{xyz}}\ dz \right)^2\ dy\ dx.$$

Third is the piece

$$J3 = \int_{x=1/2-\varepsilon}^{1/2-\varepsilon/2} \int_{y=0}^{1/2-\varepsilon} \left(\int_{z=0}^{y} F|_{A_{yzx}} + \int_{z=y}^{x} F|_{A_{zyx}} + \int_{z=x}^{1-\varepsilon-x} F|_{A_{xyz}} + \int_{z=1-\varepsilon-x}^{1-\varepsilon-y} F|_{B_{xyz}} \right.$$
$$\left. + \int_{z=1-\varepsilon-y}^{1+\varepsilon-x} F|_{C_{xyz}} + \int_{z=1+\varepsilon-x}^{3/2-x-y} F|_{T_{xyz}}\ dz \right)^2\ dy\ dx.$$

We now have dealt with all integrals involving A_{xyz}, and all remaining integrals pass through B_{zyx}. Continuing, we have

$$J4 = \int_{x=1/2-\varepsilon/2}^{1/2} \int_{y=1/2-\varepsilon}^{1-\varepsilon-x} \left(\int_{z=0}^{y} F|_{A_{yzx}} + \int_{z=y}^{1-\varepsilon-x} F|_{A_{zyx}} + \int_{z=1-\varepsilon-x}^{x} F|_{B_{zyx}} + \int_{z=x}^{1-\varepsilon-y} F|_{B_{xyz}} \right.$$
$$\left. + \int_{z=1-\varepsilon-y}^{3/2-x-y} F|_{C_{xyz}}\, dz \right)^2 dy\, dx.$$

Another component is

$$J5 = \int_{x=1/2-\varepsilon/2}^{1/2} \int_{y=0}^{1/2-\varepsilon} \left(\int_{z=0}^{y} F|_{A_{yzx}} + \int_{z=y}^{1-\varepsilon-x} F|_{A_{zyx}} \right.$$
$$\left. + \int_{z=1-\varepsilon-x}^{x} F|_{B_{zyx}} + \int_{z=x}^{1-\varepsilon-y} F|_{B_{xyz}} + \int_{z=1-\varepsilon-y}^{1+\varepsilon-x} F|_{C_{xyz}} + \int_{z=1+\varepsilon-x}^{3/2-x-y} F|_{T_{xyz}}\, dz \right)^2 dy\, dx.$$

The most complicated piece is

$$J6 = \left(\int_{x=1/2}^{2\varepsilon} \int_{y=0}^{1-\varepsilon-x} + \int_{x=2\varepsilon}^{1/2+\varepsilon/2} \int_{y=x-2\varepsilon}^{1-\varepsilon-x} \right) \left(\int_{z=0}^{y} F|_{A_{yzx}} + \int_{z=y}^{1-\varepsilon-x} F|_{A_{zyx}} + \int_{z=1-\varepsilon-x}^{x} F|_{B_{zyx}} \right.$$
$$\left. + \int_{z=x}^{1-\varepsilon-y} F|_{B_{xyz}} + \int_{z=1-\varepsilon-y}^{1+\varepsilon-x} F|_{C_{xyz}} + \int_{z=1+\varepsilon-x}^{1/2+\varepsilon} F|_{S_{xyz}} + \int_{z=1/2+\varepsilon}^{3/2-x-y} F|_{T_{xyz}}\, dz \right)^2 dy\, dx.$$

We have now exhausted C_{xyz}. The sixth piece is

$$J7 = \int_{x=2\varepsilon}^{1/2+\varepsilon/2} \int_{y=0}^{x-2\varepsilon} \left(\int_{z=0}^{y} F|_{A_{yzx}} + \int_{z=y}^{1-\varepsilon-x} F|_{A_{zyx}} + \int_{z=1-\varepsilon-x}^{x} F|_{B_{zyx}} \right.$$
$$\left. + \int_{z=x}^{1+\varepsilon-x} F|_{B_{xyz}} + \int_{z=1+\varepsilon-x}^{1-\varepsilon-y} F|_{E_{xyz}} + \int_{1-\varepsilon-y}^{1/2+\varepsilon} F|_{S_{xyz}} + \int_{1/2+\varepsilon}^{3/2-x-y} F|_{T_{xyz}}\, dz \right)^2 dy\, dx.$$

Finally, we have

$$J8 = \int_{x=1/2+\varepsilon/2}^{1-\varepsilon} \int_{y=0}^{1-\varepsilon-x} \left(\int_{z=0}^{y} F|_{A_{yzx}} + \int_{z=y}^{1-\varepsilon-x} F|_{A_{zyx}} + \int_{z=1-\varepsilon-x}^{1+\varepsilon-x} F|_{B_{zyx}} \right.$$
$$\left. + \int_{z=1+\varepsilon-x}^{x} F|_{E_{zyx}} + \int_{z=x}^{1-\varepsilon-y} F|_{E_{xyz}} + \int_{1-\varepsilon-y}^{1/2+\varepsilon} F|_{S_{xyz}} + \int_{1/2+\varepsilon}^{3/2-x-y} F|_{T_{xyz}}\, dz \right)^2 dy\, dx.$$

Adding up all of the previous integrals, and multiplying by 6, gives us the total contribution from all J-integrals, by symmetry.

4. Marginals

The marginal conditions are exactly that the following integrals must all be zero.

$$M1 = \int_{z=0}^{3/2-x-y} F|_{G_{yzx}}\ dz$$

$$M2 = \int_{z=0}^{y} F|_{G_{yzx}} + \int_{z=y}^{3/2-x-y} F|_{G_{zyx}}\ dz$$

$$M3 = \int_{z=0}^{1+\varepsilon-x} F|_{U_{yzx}} + \int_{z=1+\varepsilon-x}^{y} F|_{G_{yzx}} + \int_{z=y}^{3/2-x-y} F|_{G_{zyx}}\ dz$$

$$M4 = \int_{z=0}^{1+\varepsilon-x} F|_{U_{yzx}} + \int_{z=1+\varepsilon-x}^{3/2-x-y} F|_{G_{yzx}}\ dz$$

$$M5 = \int_{z=0}^{3/2-x-y} F|_{T_{yzx}}\ dz$$

$$M6 = \int_{z=0}^{1-\varepsilon-y} F|_{S_{yzx}} + \int_{z=1-\varepsilon-y}^{3/2-x-y} F|_{H_{yzx}}\ dz$$

$$M7 = \int_{z=0}^{1-\varepsilon-x} F|_{E_{yzx}} + \int_{z=1-\varepsilon-x}^{1-\varepsilon-y} F|_{S_{yzx}} + \int_{z=1-\varepsilon-y}^{3/2-x-y} F|_{H_{yzx}}\ dz$$

$$M8 = \int_{z=0}^{3/2-x-y} F|_{H_{yzx}}\ dz$$

5. The Computation

All polynomials have monomials with only two variables appearing. We take $F(x, y, z)$ to be a polynomial of degree 2 on each of $A_{xyz}, B_{xyz}, C_{xyz}$, and use degree 5 on the other six relevant components of R. Optimizing the resulting quadratic form and taking $\varepsilon = 1/4$, we get $M \geq 2.0009$.