SOME NOTES ON A VARIATIONAL PROBLEM

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Let $1/4 \leqslant \varepsilon \leqslant 1/2$. All variables x, y, z are assumed to be non-negative by default. Let $M = \hat{M}''_{3,\varepsilon,1}$ be the best constant for which one has the inequality

$$3 \int\int_{x+y \leqslant 1-\varepsilon} (\int F(x,y,z)\ dz)^2\ dxdy \leqslant M \int_R F(x,y,z)^2\ dxdydz \tag{0.1}$$

whenever F is supported on $R := \{(x,y,z) : x+y+z \leqslant 3/2\}$, is symmetric, and obeys the vanishing marginal condition

$$\int F(x,y,z)\ dz = 0 \text{ whenever } x+y \geqslant 1+\varepsilon. \tag{0.2}$$

Proposition 0.1 (Adjoint formulation)**.** *M is also the best constant for which one has the inequality*

$$\int_R (G(x,y) + G(y,z) + G(x,z))^2\ dxdydz \leqslant 3M \int\int_{x+y \leqslant 1-\varepsilon} G(x,y)^2\ dxdy \tag{0.3}$$

whenever G is symmetric, supported on $\{(x,y) : x+y \leqslant 1-\varepsilon\} \cup \{(x,y) : 1+\varepsilon \leqslant x+y \leqslant 3/2\}$, and the function $F(x,y,z) := G(x,y) + G(y,z) + G(x,z)$ obeys the vanishing marginal condition (0.2)*.*

Proof. Let us first show that if M is such that (0.1) holds, then (0.3) also holds. Let G be as in (0.3), and set $F(x,y,z) := G(x,y) + G(y,z) + G(x,z)$. Then by symmetry,

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Fubini, (0.2), the support of G, Cauchy-Schwarz, and (0.1), one has

$$\begin{aligned}
\int_R &(G(x,y)+G(y,z)+G(x,z))^2\ dxdydz \\
&= \int_R (G(x,y)+G(y,z)+G(x,z))F(x,y,z)\ dxdydz \\
&= 3\int_R G(x,y)F(x,y,z)\ dxdydz \\
&= 3\int_{x+y\leqslant 3/2} G(x,y)(\int F(x,y,z)\ dz)dxdy \\
&= 3\int_{x+y\leqslant 1+\varepsilon} G(x,y)(\int F(x,y,z)\ dz)dxdy \\
&= 3\int_{x+y\leqslant 1-\varepsilon} G(x,y)(\int F(x,y,z)\ dz)dxdy \\
&\leqslant (3\int_{x+y\leqslant 1-\varepsilon} G(x,y)^2\ dxdy)^{1/2}(3\int_{x+y\leqslant 1-\varepsilon}(\int F(x,y,z)\ dz)^2\ dxdy)^{1/2} \\
&\leqslant (3\int_{x+y\leqslant 1-\varepsilon} G(x,y)^2\ dxdy)^{1/2}(M\int_R F(x,y,z)^2\ dxdydz)^{1/2} \\
&= (3\int_{x+y\leqslant 1-\varepsilon} G(x,y)^2\ dxdy)^{1/2}(M\int_R (G(x,y)+G(y,z)+G(x,z))^2\ dxdydz)^{1/2},
\end{aligned}$$

and (0.3) follows.

Conversely, suppose that M is such that (0.3) holds; we now show (0.1). Let F be as in (0.1), and let G_0 be defined by setting $G_0(x,y) := \int F(x,y,z)\ dz$ when $x+y \leqslant 1-\varepsilon$, and $G_0(x,y) = 0$ otherwise. Let G_1 be a symmetric function supported on $\{1+\varepsilon \leqslant x+y \leqslant 3/2\}$ to be chosen later, and set $G := G_0+G_1$ and $\tilde F(x,y,z) := G(x,y)+G(y,z)+G(x,z)$. We can choose G_1 so that the function $\tilde F$ obeys the vanishing marginal condition (0.2); indeed, in the symmetric functions in $L^2(R)$, one may verify that the class of F obeying (0.2) is nothing other than the orthogonal complement of the functions of the form $G_1(x,y)+G_1(y,z)+G_1(x,z)$ with G_1 symmetric and supported on $\{1+\varepsilon \leqslant x+y \leqslant 3/2\}$.

By (0.2), symmetry, Cauchy-Schwarz, and (0.3) one has

$$\begin{aligned}
3\int\int_{x+y\leqslant 1-\varepsilon} &(\int F(x,y,z)\ dz)^2\ dxdy \\
&= 3\int\int_{x+y\leqslant 1-\varepsilon}(\int F(x,y,z)\ dz)G_0(x,y)\ dxdy \\
&= 3\int_R F(x,y,z)G_0(x,y)\ dxdydz \\
&= 3\int_R F(x,y,z)G(x,y)\ dxdydz \\
&= \int_R F(x,y,z)(G(x,y)+G(y,z)+G(x,z))\ dxdydz \\
&\leqslant (\int_R F(x,y,z)^2\ dxdydz)^{1/2}(\int_R (G(x,y)+G(y,z)+G(x,z)^2\ dxdydz)^{1/2} \\
&\leqslant (\int_R F(x,y,z)^2\ dxdydz)^{1/2}(3M\int_{x+y\leqslant 1-\varepsilon} G(x,y)^2\ dxdy)^{1/2} \\
&= (\int_R F(x,y,z)^2\ dxdydz)^{1/2}(3M\int_{x+y\leqslant 1-\varepsilon}(\int F(x,y,z)\ dz)^2\ dxdy)^{1/2}
\end{aligned}$$

and (0.1) follows. □

We can simplify the left-hand side of (0.3) further, using the vanishing marginal condition (0.2). Introduce the inner product

$$\langle G, G'\rangle := \int_R (G(x,y)+G(y,z)+G(x,z))(G'(x,y)+G'(y,z)+G'(x,z))\ dxdydz$$

then the left-hand side of (0.3) is

$$\langle G, G\rangle = \langle G, G_1\rangle + \langle G, G_0\rangle.$$

On the other hand, the vanishing marginal condition (0.2) tells us that $\langle G, G_1\rangle = 0$. Thus

$$\langle G, G\rangle = \langle G, G_0\rangle.$$

We thus see that

$$M = \sup \frac{\langle G, G_0\rangle}{3\int\int G_0^2\ dxdy}$$

whenever G_0, G_1 are symmetric and supported on $\{x+y\leqslant 1-\varepsilon\}$ and $\{1+\varepsilon\leqslant x+y\leqslant 3/2\}$ respectively, and the function

$$F(x,y,z) = G(x,y) + G(y,z) + G(x,z)$$

with $G = G_0 + G_1$ obeys (0.2).

We simplify things further. Observe that

$$\begin{aligned}
\langle G, G_0\rangle &= \int_R (G(x,y)+G(y,z)+G(x,z))(G_0(x,y)+G_0(y,z)+G_0(x,z))\ dxdydz \\
&= 3\int_R (G(x,y)+G(y,z)+G(z,x))G_0(x,y)\ dxdydz \\
&= 3\int_R G_0(x,y)^2\ dxdydz + 6\int_R G(x,z)G_0(x,y)\ dxdydz \\
&= 3\int_{x+y\leqslant 1-\varepsilon} G_0(x,y)^2(3/2-x-y)\ dxdy + 6\int_{x+y\leqslant 1-\varepsilon} G_0(x,y)(\int_0^{3/2-x-y} G(x,z)\ dz)dxdy \\
&= 3\int_{x+y\leqslant 1-\varepsilon} G_0(x,y)^2(3/2-x-y)\ dxdy + 6\int_{x+y\leqslant 1-\varepsilon} G_0(x,y)(\int_0^{1-\varepsilon-x} G_0(x,z)\ dz)dxdy \\
&\quad + 6\int_{x+y\leqslant 1-\varepsilon; y\leqslant 1/2-\varepsilon} G_0(x,y)(\int_{1+\varepsilon-x}^{3/2-x-y} G_1(z,x)\ dz)dxdy \\
&= 3\int_{x+y\leqslant 1-\varepsilon} G_0(x,y)^2(3/2-x-y)\ dxdy + 6\int_{x\leqslant 1-\varepsilon}(\int_0^{1-\varepsilon-x} G_0(x,y)\ dy)^2 dx \\
&\quad + 6\int_{1+\varepsilon\leqslant x+z\leqslant 3/2} G_1(x,z)(\int_{y\leqslant 3/2-x-z} G_0(x,y)\ dy)\ dxdz.
\end{aligned}$$

Here we use that $\varepsilon \geqslant 1/4$ to ensure that $1-\varepsilon-x \leqslant 3/2-x-y$ whenever $x+y\leqslant 1-\varepsilon$. Thus

$$M = \sup \frac{J}{I}$$

where

$$\begin{aligned}
J := &\int_{x+y\leqslant 1-\varepsilon} G_0(x,y)^2(3/2-x-y)\ dxdy + 2\int_{x\leqslant 1-\varepsilon}(\int_0^{1-\varepsilon-x} G_0(x,y)\ dy)^2 dx \\
&+ 2\int_{1+\varepsilon\leqslant x+z\leqslant 3/2} G_1(x,z)(\int_{y\leqslant 3/2-x-z} G_0(x,y)\ dy)\ dxdz
\end{aligned}$$

and

$$I := \int\int G_0^2\ dxdy.$$

Now we work on G_1. From (0.2) we see that

$$\int_0^{3/2-x-y} G(x,y)+G(z,y)+G(x,z)\ dz = 0$$

whenever $x+y\geqslant 1+\varepsilon$, thus

$$G_1(x,y)(3/2-x-y) + \int_0^{3/2-x-y} G(z,y)\ dz + \int_0^{3/2-x-y} G(x,z)\ dz = 0. \tag{0.4}$$

At this point, we set $\varepsilon = 1/4$, and introduce the large triangular region

$$A := \{x+y\leqslant 3/4\}$$

(that G_0 is supported on), and the smaller triangular and trapezoidal regions

$$\begin{aligned} B &:= \{x + y \leqslant 3/2; y \geqslant 5/4\} \\ C &:= \{x + y \geqslant 5/4; y \leqslant 5/4; x \leqslant 1/4\} \\ D &:= \{5/4 \leqslant x + y \leqslant 3/2; x \geqslant 1/4; y \geqslant 3/4\} \\ E &:= \{x + y \geqslant 5/4; x, y \leqslant 3/4\} \end{aligned}$$

which cover a bit more than half of the support of G_1. Let $G_{1,B}, G_{1,C}, G_{1,D}, G_{1,E}$ be the restrictions of G_1 to B, C, D, E respectively; we assume these to be polynomial, as we do with G_0. We also introduce the average

$$H(x, w) := \frac{1}{w} \int_0^w G_0(x, y)\ dy \tag{0.5}$$

of G_0; note that H is still a polynomial on A if G_0 is, although H is no longer symmetric. The vanishing moment condition (0.4) then becomes

$$(3/2-x-y)G_{1,B}(x, y)+\int_0^{3/2-x-y} G_{1,B}(z, y)\ dz+(3/2-x-y)H(x, 3/2-x-y) = 0 \tag{0.6}$$

on B,

$$(3/2-x-y)G_{1,C}(x, y)+\int_{5/4-y}^{3/2-x-y} G_{1,C}(z, y)\ dz+(3/2-x-y)H(x, 3/2-x-y) = 0 \tag{0.7}$$

on C,

$$(3/2 - x - y)G_{1,D}(x, y) + 0 + (3/2 - x - y)H(x, 3/2 - x - y) = 0 \tag{0.8}$$

on D, and

$$(3/2-x-y)G_{1,E}(x, y)+(3/2-x-y)H(y, 3/4-y)+(3/2-x-y)H(x, 3/4-x) = 0 \tag{0.9}$$

on E. Thus we have exact formulae

$$G_{1,D}(x, y) = -H(x, 3/2 - x - y) \tag{0.10}$$

and

$$G_{1,E}(x, y) = -H(y, 3/4 - y) - H(x, 3/4 - x) \tag{0.11}$$

on D, E respectively, and linear constraints connecting the coefficients of $G_{1,B}$, $G_{1,C}$ with the coefficients of H (which are defined in terms of G_0).

We then have

$$J = \int_{x+y\leqslant 3/4} G_0(x,y)^2(3/2-x-y)\ dxdy$$
$$+ 2\int_{x\leqslant 3/4} (\int_0^{3/4-x} G_0(x,y)\ dy)^2 dx$$
$$+ 2\int_B G_{1,B}(x,z)(3/2-x-z)H(x,3/2-x-z)\ dxdz$$
$$+ 2\int_C G_{1,C}(x,z)(3/2-x-z)H(x,3/2-x-z)\ dxdz$$
$$+ 2\int_D G_{1,D}(x,z)(3/2-x-z)H(x,3/2-x-z)\ dxdz$$
$$+ 2\int_E G_{1,E}(x,z)(3/4-x)H(x,3/4-x)\ dxdz$$

and

$$I = \int_{x+y\leqslant 3/4} G_0(x,y)^2\ dxdy.$$

We thus have

$$M = \sup \frac{J}{I}$$

where $G_0, G_{1,B}, G_{1,C}$ range over all polynomials of two variables obeying the constraints (0.6), (0.7), where $H, G_{1,D}, G_{1,E}$ are defined by (0.5), (0.10), (0.11), with G_0 symmetric. If we make $G_0, G_{1,B}, G_{1,C}$ be of degree up to D, then there are about $5D^2/2$ degrees of freedom with $2D^2$ constraints.

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